The Spirit of Mathematics

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Introduction

These dialogues have a vastly ambitious theme, encompassing the role of ancient Greek civilization and culture in the evolution of the modern world. Within this all-embracing aim, I will be focusing on mathematics, one of the oldest and most pervasive aspects of civilization. It also happens to be my own field, and while this enables me to write with some confidence on the content, I may be guilty of overemphasizing its influence on other fields. However, I do believe that mathematics has always played a fundamental part in the development of civilization, especially in ancient Greece and in our own times.

Let me begin by giving my personal and somewhat simplified view of where mathematics fitted into Greek culture. There is little doubt that philosophy, in the broad sense, was at the centre of Greek thought. We can diagrammatically link it to its various neighbours or offspring, with key names as exemplars (Figure 1).

![Figure 1. Philosophy in Ancient Greece](image)

For Plato and his school, Mathematics meant geometry, but it also represented precise thinking, with the key concept of proof. This has become the hallmark of mathematics ever since, and gets us as near to certainty as is possible.

In the diagram I have placed Art and Natural Philosophy (or science) in adjacent places, because of their close links with mathematics. This is clear and well understood in the case of science, but not so well recognized in the case of art. But
sculpture and architecture, both of which reached new heights in ancient Greece, are embodiments of geometry and mathematical principles.

To Plato there would have been no problem in associating mathematics with moral philosophy and the problems of ethics. Precise thinking and the constant questioning of basic assumptions, which underlie the Socratic dialogues, are the characteristics of mathematics.

In the following pages I will examine the various links implicit in my diagram in more detail, but perhaps I should first digress to explain my own view of the nature of mathematics. I have already described it as consisting of precise thought. This may describe its process, but how about its content? There is little doubt that mathematics has emerged from our experience of the natural world, either visually, in geometry, or discretely, by enumeration and counting. But mankind has done more than just observe: it has built internal concepts and structures that reflect what it has seen. Analogy and abstraction are the key ideas involved, and they remain as pertinent now as two thousand years ago. The whole impressive edifice of mathematics is a human creation built on observation of the natural world.

**History and Geography**

Having started with Greek philosophy and its close relationship with mathematics, let me now turn to the historical perspective. Again I will indulge in the mathematical predilection for diagrams by representing more than two millennia of cultural transmission in an oversimplified picture (Figure 2).
While all the civilizations and eras depicted in this diagram have been influenced in many ways by their predecessors, mathematics has been a key component. Its universal nature, its intellectual structure, and its practical applications have ensured that the ideas moved across frontiers both physical and cultural.

Until recently, most scholars brought up in the classical western tradition emphasized the debt that the modern world owed to ancient Greece, while acknowledging \textit{en passant} that the Arabs had conserved and transmitted Greek culture. But it is now increasingly realized that the influx of older ideas from further east was of great importance. The Arab contribution has also been reassessed and seen to be much more than one of transmission. The creation of algebra by the Arabs has always been recognized, the word itself being of Arab origin. But the sophistication of Arab mathematics and astronomy is only now being realized.

Take the case of the twelfth-century Persian poet and astronomer Omar Khayyam. Not only did he write a treatise on the cubic equation long before the Italian mathematicians, but he measured the sidereal year with greater accuracy than was achieved, four centuries later, by the western astronomers who produced the famous Gregorian calendar.

In other words, mathematics marched on, making continuous progress from the ancient Greeks through the great Arab civilization and then back into Europe via
Spain and Italy. Moreover, its earlier roots in Mesopotamia and the Far East take us back to the dawn of civilization.

Mathematics and Philosophy

Let us now pursue in more detail the close links between mathematics and philosophy that I described briefly for ancient Greece.

“Precise thinking” was the term I used, and it can also be identified with “logic.” As such it has played an important part in the foundations of both disciplines, with notable figures such as Descartes, Leibniz, Bertrand Russell, and Gödel. Even though “ultimate certainty” has proved elusive, much has been learnt on the way. Perhaps the most influential work in this area was that of Alan Turing, which led to the development of the digital computer. It is a striking example of fundamental logical thinking going back to Plato and now, combined with modern technology, underpinning the entire world economy.

I will shortly turn to the links between mathematics and art, but as a pointer in that direction I should draw attention to Lewis Carroll, the creator of Alice in Wonderland. Although millions round the world are familiar with this imaginative story, not many realize that Carroll was a mathematician whose speciality was logic. In fact a careful reading of Alice exhibits the striking role of paradox and contradiction. Logic here takes art by surprise.

Mathematics and Art

Of all the arts, the closest to mathematics is undoubtedly architecture. Not only are buildings three-dimensional constructions which have to conform to the rules of geometry, but architecture provides an excellent template to explain the nature of mathematics to a lay audience. In both disciplines there is a fusion of vision and technique. Buildings, like theorems, must rest on secure foundations, but they should also please the eye and lift the soul. They also serve society in practical ways, and the crucial relation between pure and applied mathematics has its counterpart in the tension between form and function in architecture. Finally, over the centuries both mathematics and architecture have responded to developments in technology, whether it be with materials or with ideas. Algebra and calculus provide opportunity and challenge as clearly as glass or concrete.
While the Greeks sought beauty both in buildings and in the human form, subsequent civilizations dominated by religion found different solutions. Christianity erected monumental cathedrals, with iconic paintings emphasizing spiritual rather than physical humanity (although St. George killing the dragon attempts to combine both). On the other hand the Islamic world eschewed the human body almost entirely, but created mosques which in their simplicity and grandeur rival churches and cathedrals. There is mathematics in both styles, but the emphasis is different, with the dome replacing the tower and the minaret replacing the steeple. The fact that architecture and mathematics transcend religious differences is significantly demonstrated at the two extremes of the Mediterranean. In Constantinople, Hagia Sophia became a great mosque, while in Cordova an incredible mosque actually houses a church in its interior. Fortunately, in both cases, religious fervour was kept under control and not allowed to destroy the artistic masterpieces of the preceding civilization. But even if vandalism had triumphed, and the buildings been reduced to rubble, the mathematical and aesthetic ideas would have lived on. The pen is mightier than the sword and mathematics lives by the pen: writing formulae rather than words.

Islamic art is in a sense more mathematical than Christian art, as geometrical patterns and motifs replace human figures. This is evident in the great variety of decorative designs on the walls of mosques. The symmetries that were not formalized and identified by mathematicians until centuries later are all to be found in the old Islamic world. The Alhambra is a pictorial library for the student of symmetry.

While I have focused on architecture, mathematics also plays an important part in both painting and music. The discovery of perspective, leading to more convincing paintings of three-dimensional subjects, was a turning point at the beginning of the Renaissance. Painting could now compete in depth with architecture, as demonstrated by Michelangelo, the great master of both.

The role of mathematics in music is equally fundamental, though played out in time rather than in space. The mathematical basis of musical notes was already recognized by Pythagoras, and continues to constrain and challenge musical composers and performers up to the present time. Through music mathematics is
also brought close to poetry, as evidenced by Omar Khayyam and, in a different way, by Lewis Carroll.

While the general public can understand the role of mathematics in the various arts, as I have briefly described, it is much harder to appreciate the reverse: the role of beauty in mathematics. Most people studied mathematics to various levels at school, found the subject difficult, and were happy in due course to leave it behind. Their memory of it is usually one of struggle rather than enjoyment, and few can see through the technicalities to the underlying beauty of the ideas. But we professional mathematicians know that our subject is profoundly beautiful, the structure of ideas and arguments akin to a soaring medieval cathedral, with the detailed craftsmanship an essential component.

The great German mathematician Hermann Weyl said that, in all his work, he strove for beauty and truth, but when in doubt he usually chose beauty. This statement sounds perverse. For a mathematician, the search for the ultimate truths, demonstrated by proof, is the Holy Grail. How can aesthetic judgement compete? Moreover beauty is subjective, “in the eye of the beholder,” while truth is objective and recognized by all. Surely, you will say, Weyl was joking and deliberately provocative. In fact I am sure he was totally serious in identifying an essential aspect not only of mathematics, but of all human thought.

I will paraphrase Weyl by saying that for us mathematicians “we search for truth, but our guiding light is beauty”. As a working mathematician I only dimly perceive “the ultimate truth,” but I can immediately appreciate beauty: I know it when I see it. I grasp it, follow it, and hope it leads in the right direction.

Even if one concedes this view, how can one go further, with Weyl, and prefer beauty when it conflicts with truth? The simple answer is that such contradictions may only be apparent: what is deemed to be true may be provisional knowledge that has to be modified. Beauty, on the other hand, carries immediate conviction. It is personal and varies with the individual, but if we each pursue our own path, one of us is more likely to find the truth.

There are many famous examples which show that beauty has in the end triumphed over contradiction with apparent truths. A notable case was in Weyl’s own work, when in the aftermath of Einstein’s general theory of relativity he attempted to incorporate Maxwell’s theory of electromagnetism in geometric form.
Dismissed as physically wrong by Einstein, it survived, got modified, and is now universally accepted as the basis of modern physics.

**Mathematics and Natural Science**

The connection between mathematics and natural science, especially physics, is so deep and so well known that it is unnecessary to dwell too long on it. I will however pick out a few key themes.

The first is the key role played by a few fundamental equations, of which the prototype was Newton’s inverse-square law of gravitation. This emerged in three steps. First, there was the new technology of grinding lenses, which produced the telescope and led Galileo to make his astronomical observations. Second was the accumulation and interpretation of data that led eventually to Kepler’s laws of planetary motion. Finally, there was the grand synthesis by Isaac Newton, who developed the calculus that enabled him to establish a convincing mathematical theory.

The idea that an inverse-square law might explain why the planets orbited the sun had been around for some time and had been put forward by others, such as Robert Hooke. But it was Newton who had the mastery of mathematics that enabled him to go from the equation to the elliptical planetary orbits.

This shows that a single equation can encapsulate a whole theory. Physical insight and observation using new technology provides the background from which the equation emerges, but a whole mathematical corpus, the calculus, is then needed to extract the consequences. Similar stories hold for a number of other fundamental equations, notably including Maxwell’s equations of electromagnetism, Einstein’s equation of general relativity, and Schrödinger’s equation of quantum mechanics.

It is quite remarkable how much science is contained in these few mathematical equations. Here is mathematical beauty at its most convincing. Physicists call these “laws of nature,” but this raises deep philosophical questions. Who lays down the laws? Why does nature embody such beautifully simple laws? Do the laws represent “reality,” or are they an artefact of the human mind: the way we see and understand nature?
What we can say is that scientists of all types believe in the existence and reality of such laws. They believe that the natural world is indeed built on coherent principles, and that it is the task of the scientist to discover these laws. In the end we scientists believe in a rational world as an article of faith. For some it is evidence of divine creation. For others it remains a mystery, but such faith appears to have been justified. The success of science is difficult to deny, though one may continue to speculate on the “ultimate reality.” In physics, the belief is that all laws can be expressed by fundamental equations. The power of mathematics is precisely that it provides a framework in which simple equations can explain and describe a vast range of physical phenomena. An equation is like a seed which can germinate and produce a giant tree.

Here I have, for the purpose of analogy, crossed over from physics to biology. Insofar as biology relies on physics and chemistry, it too rests on mathematical foundations. But at the more complex level of molecular biology, genetics, and evolution, it is not yet clear how much impact mathematical ideas will have. I will return to this in subsequent sections.

**Probability and Complexity**

Once we move from basic physics into more complex subjects, including chemistry, biology, and economics, the nature of mathematical formulation and laws changes. The focus of interest moves from a single small, perhaps idealized particle to large numbers of such particles. Average properties and probabilities take over. New mathematics is developed to provide the right language and technique. But key mathematical principles survive. Analogy, abstraction, formulae, and proof all remain, even though they are now applied to quite different objects. This demonstrates the flexibility of mathematics, its ability to provide a precise framework for our new situations, with old lessons, ideas, and techniques providing inspiration for entirely new fields. For example, the random “Brownian motion” of pollen seeds in liquids is governed by the same equations as the vagaries of the financial markets.

If the laws of physics were already there at the “Big Bang” and have guided the subsequent evolution of the cosmos, the role of mankind has been limited to that of an observer, with the mathematician acting as scribe. But the laws of economics are
of a different nature: mankind, in the form of organized society, is now in the arena, and mathematical theory becomes a form of introversion: man studying himself.

But God and mathematics make no distinction, and lay down laws or equations both for galaxies and for human society. The spirit of mathematics pervades the universe, whether in theological or scientific form.

Brain and Mind

Nowhere is the introverted nature of human thought more evident than in the study of brain and mind. The domain of philosophy in earlier times, it has now moved to the forefront of biology and psychology. Much has been learnt scientifically in recent decades, and it is clear that this whole area will be central to the science of the twenty-first century.

Does mathematics have a role to play in this evolving discipline? Can mathematicians flex their muscles here as effectively as they have done in the physical sciences? Will the equation have met its match in the very place of its birth? I am sure that Plato would have delighted in such a philosophical and speculative discussion.

I raise these questions in an entirely open way. I am not one of those who believe that mathematics is the key to all knowledge and that its past success is a guarantee to a similar future. We should survey the scene, examine the evidence, and identify promising directions where mathematics may have an impact. We may be able to help our fellow scientists, but time alone will tell whether we become key players, or only play minor parts as technical assistants.

History has shown that new scientific challenges encourage the development of new mathematical theories, where both the concepts and the techniques emerge in a natural way. But the universality and coherence of mathematics means that old ideas get adapted to new situations, so that the calculus for instance can be applied well beyond the theory of planetary motion. What the mathematician has to offer is a flexible mind with access to ancient wisdom.

Philosophers and scientists have long argued about the distinction between brain and mind, between the physiology and the psychology, between the detailed mechanisms and the final output. As our knowledge advances the frontiers shift, distinctions blur, and yesterday’s questions seem artificial or ill-posed.
Biologists no longer ask: what is life? With the understanding of DNA, viruses, and genetics, we now know too much for such a simplistic question. The big questions on mind will undoubtedly suffer a similar fate.

As a mathematician—what you might call a professional thinker—my favourite question is: what is thought? Sadly this question will in the end probably disappear, as being too naïve, but in our present state of knowledge it is worth asking and may point us in a profitable direction.

My friend, the distinguished neurophysiologist Semir Zeki, has pointed out that the process of abstraction is fundamental for all thought. The realization that a chair, even if seen from all angles, is a single entity and embodies the abstract philosophical notion of “chairness,” is a key step in our making sense of the external world. While lower forms of life must also have passed this test in order to survive, Homo sapiens has climbed much further up the ladder of abstraction. Mathematicians in particular have specialized in the art or science of abstraction and pushed it to extremes.

Abstraction, whether in the hidden reaches of human thought or in the more formal apparatus of the mathematician, builds on itself. I described it as a ladder, and this illustrates its hierarchical nature. In mathematics this is well understood and embedded in its history. Chairs, or more appetisingly oranges and apples, acquire numbers, which algebra then turns into unknown variables $x, y, ...$. While initially these letters stand for numbers, in higher algebra they acquire an independent status, standing for nothing else. What become important now are the mutual relations between the symbols, expressed sometimes by equations. At the next level, these relations or equations are themselves given independent status. Algebraists manipulate the symbols much as we handle pound notes (which once represented gold), or as modern bankers handle complex financial instruments called “derivatives.” Such abstraction, built in hierarchical levels, is the secret of success both in mathematics and (hopefully) in the financial world. It provides a framework in which complex ideas can be handled according to precise rules.

It seems that some such hierarchical process of abstraction must underlie human thought. We are a long way from understanding how this is achieved, but it would be surprising if mathematicians could not shed light on the process. There are many tools in the mathematical kit-box, some more fully developed than others,
depending one might say on market forces. Applied mathematicians make tools on demand, fashioned for the client’s needs, while pure mathematicians make tools “just for fun.” The abstract tools that the cognitive scientists of the future will need may or may not already be in the mathematician’s tool kit, but mathematicians can respond to new demands and provide new tools for their scientific colleagues.

**Codes**

Let me conclude by discussing the topic of codes and ciphers, in which mathematicians have played an important part from ancient times. The use of disguised messages which can only be read by the sender and the intended recipient must, in various ways, have gone hand-in-hand with the development of symbolic writing. Certainly it was extensively used in the English Civil War of the seventeenth century, and the famous mathematician John Wallis was a master of secret codes. In fact, depending on the fortunes of war, Wallis was employed first by one side and then by the other. His mathematical skills made him too valuable to be executed.

It is easy to see why mathematicians should be adept at both encrypting and deciphering secret codes. The use and permutation of symbols is what algebra is all about. But there is more to it than just the use of symbols. Grammar and syntax are also common to both languages and mathematics. Wallis took a keen interest in grammar, and wrote the first textbook on the subject. This may have assisted his entry into the cloak-and-dagger world of the Civil War.

In more recent times mathematicians have been in the forefront of those working on codes. During the 1939–1945 war, the British analysis of German ciphers was centred at Bletchley Park, where the cream of the mathematical establishment was instrumental in breaking German codes. This team effort was under the leadership of Alan Turing, whose pioneering work on the modern computer I have already alluded to. Codes and computers go naturally together.

While codes remain vital to the military all over the world, they are now widely used by banks to provide security for financial transactions. Interestingly, prime numbers (such as 3, 5, 7, 11, ...) play a vital part in such coding. In modern biology, one of the great advances was the discovery of the genetic code, and this provides
one natural link with mathematics. Following genes back in evolution to understand our ancestry (and that of animal species) is fertile ground for mathematicians.

The role of prime numbers in coding theory shows the unexpected way in which pure mathematics developed “just for fun” or for the “glory of God,” can be put to use many decades later. The famous English mathematician G. H. Hardy was a number theorist and an extreme pure mathematician. He prided himself on never having done anything “useful.” Actually his abhorrence of useful things had a humane side: he was a pacifist, and by “useful” he meant something that could be put to military use to kill people. So it is ironic that prime numbers, the love of Hardy’s life, and the purest form of mathematics imaginable, should many years later be put to use by the military. Hardy must be turning in his grave!

This brings me finally to the question of ethics in science. Mathematicians and scientists devote their working lives to the pursuit of truth and its use for the benefit of mankind. Academies of science usually have phrases to this effect in their founding documents. Unfortunately knowledge can be misused, with disastrous consequences. The most spectacular case is that of nuclear weapons, the fruit of basic scientific research, but employed in Japan in 1945 and now stockpiled in vast numbers that hang as the sword of Damocles over our heads.

Many scientists, myself included, have spent years trying to prevent the ultimate catastrophe of nuclear war. Fortunately the politicians are finally beginning to listen, and sanity has appeared on the scene. Hope is in the air.

Ethical questions, beyond nuclear weapons, have come increasingly to the fore as science makes a bigger and bigger impact on our lives. Pandora’s Box is well and truly open, and the challenge of making scientific progress while avoiding disastrous consequences is the fundamental issue for mankind at the present time, and for the foreseeable future. Since these dialogues cover the whole range of scholarly discourse, they provide a suitable forum for the great dilemma posed—metaphorically and literally—by the “explosion of knowledge.”